Smooth Calabi-Yau varieties with large index and Betti numbers

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Results

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Definition

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Definition

The *index* of a Calabi-Yau variety is the smallest positive integer m so that $mK_X \sim 0$.

Definition (Sylvester's sequence)

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$$s_{n+1} = s_0 \cdots s_n + 1$$

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n	s_n
0	2
1	3
2	7
3	43
4	1807
5	3263443
6	10650056950807

Theorem (Smooth Calabi-Yau varieties with large index)

For every $n \geq 1$, there exists a smooth, projective Calabi-Yau n-fold $V^{(n)}$ with index $(s_{n-1} - 1)(2s_{n-1} - 3)$.

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n	$\mathrm{index}(V^{(n)})$
1	1
2	6
3	66
4	3486
5	6521466
6	21300104111286
7	226847426110811738551148466

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3. Find a crepant, projective, μ_m-equivariant resolution[1] $\widetilde{X} \longrightarrow X$.

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5. Form the variety $V^{(n)} = \frac{\widetilde{X} \times E}{\mu_m}$.

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Dim 3. $X = \widetilde{X}$ the K3 surface with a non-symplectic automorphism of order 66.

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Remarks

1. Esser, Totaro, and Wang posed the same conjecture for the terminal Calabi-Yau varieties they constructed.
2. It's unknown if there is *any* uniform upper bound for the indices of Calabi-Yau varieties.

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n	$\sum_{i=0}^{2n} b_i(W^{(n)})$	$\chi(W^{(n)})$
1	4	0
2	24	24
3	1008	-960
4	1820448	1820448
5	5940926462016	-5940922821120
6	63271205161020798539584896	63271205161020798539584896

Esser, Totaro, and Wang proved that the hypersurface $X \subseteq \mathbb{P}$ in the construction of $V^{(n)} = \frac{\widetilde{X} \times E}{\mu_m}$ has these exact values as its *orbifold* Betti numbers (defined by Chen and Ruan[2]).

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Hence, we take $W^{(n-1)} = \widetilde{X}$.

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$\mathbb{C}^4/\mu_2$, for instance, is a Gorenstein quotient singularity with no crepant resolution.

Background on toric varieties

A *toric variety* is given by the data of a lattice N and a rational polyhedral fan Δ in $N \otimes \mathbb{R}$.

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See Fulton's *Introduction to Toric Varieties* for more details.

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Then $X(\Delta) = \mathbb{P}^n$.

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$$t(x_0, \ldots, x_n) = (t^{a_0} x_0, \ldots, t^{a_n} x_n).$$

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$$P = \mathrm{Conv}(e_0, \ldots, e_{n-1}, -(a_0, \ldots, -a_{n-1})) \subseteq \mathbb{R}^n.$$

Lemma

$X(\Delta)$ is smooth if and only if every (maximal) cone is generated by part of an integral basis for the lattice N.

Every ray $\rho \in \Delta$ determines a torus-invariant irreducible divisor $D_\rho = V(\rho)$.

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For instance, the canonical divisor is given by

$$K_{X(\Delta)} = -\sum_\rho D_\rho.$$

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$$\psi_D(n_\rho) = -a_\rho$$

where n_ρ is the smallest nonzero lattice point on the ray ρ.

Lemma

D is ample if and only if ψ_D is strictly convex.

Let $(N, \Delta), (N', \Delta')$ be lattices with fans, and let $f : N \longrightarrow N'$ be a morphism so that for every $\sigma \in \Delta$ there is a $\sigma' \in \Delta'$ so that $f(\sigma) \subseteq \sigma'$.

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Lemma

Let $f : (N, \Delta) \longrightarrow (N', \Delta')$ as before. Let D' be a torus-invariant Cartier divisor on $X(\Delta')$ with support function $\psi_{D'}$. Then the support function of $\phi^* D$ is $\psi_{D'} \circ \phi$.

The weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ corresponds to the polytope $P = \mathrm{Conv}(\overline{e_0}, \ldots, \overline{e_n})$ by forming the fan Δ_P as before.

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If we have a triangulation $\mathcal{T}$ of P, we can form a new fan $\Delta_{\mathcal{T}}$ generated by the elements of $\mathcal{T}$.

- The identity $\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ sends $\Delta_{\mathcal{T}}$ to Δ_P and hence induces a birational morphism $\pi : X(\Delta_{\mathcal{T}}) \longrightarrow X(\Delta_P) = \mathbb{P}(a_0, \ldots, a_n)$.

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- $X(\Delta_{\mathcal{T}})$ is smooth if and only if for every $\sigma \in \mathcal{T}$ of dimension n (which we refer to as a *cell* of $\mathcal{T}$), the nonzero vertices of σ form an integral basis of $\mathbb{Z}^n$, i.e. are *unimodular simplices*. We say $\mathcal{T}$ is *unimodular*.

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- $X(\Delta_{\mathcal{T}})$ is projective if and only if there is a continuous, piecewise $\mathbb{Z}$-linear function $P \longrightarrow \mathbb{R}$ which is strictly convex with respect to $\mathcal{T}$, i.e. the domains of linearity are precisely the cells of $\mathcal{T}$. We say $\mathcal{T}$ is *regular*.

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- $\pi^* K_{\mathbb{P}} = K_{X(\Delta_{\mathcal{T}})}$ as π is induced by the identity.

So to find a toric, projective, crepant resolution of a weighted projective space $\mathbb{P}$ we need to find a regular, unimodular triangulation of the corresponding polytope P.

Triangles

Let

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$$w_1^{(n)} = \left(-\frac{2s_{n-1}-2}{s_0}, \dots, -\frac{2s_{n-1}-2}{s_{n-2}}, -1\right)$$
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and let

$$P_1^{(n)} = \mathrm{Conv}(e_0, \ldots, e_{n-1}, w_1^{(n)})$$
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$$\mathbb{P}_1^{(n)} = \mathbb{P}\left(\frac{2s_{n-1}-2}{s_0}, \dots, \frac{2s_{n-1}-2}{s_{n-2}}, 1, 1\right)$$
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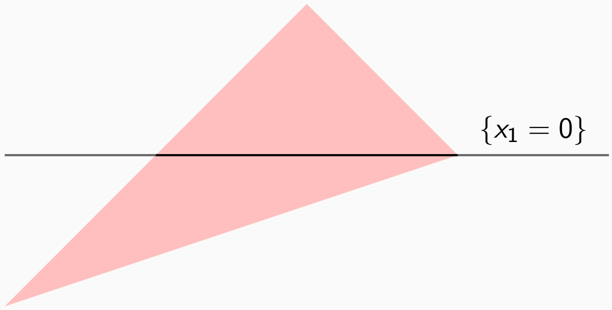
Hence, we must find a regular, unimodular triangulation of $P_1^{(n)}$.

Figure 1: $P_1^{(2)}$



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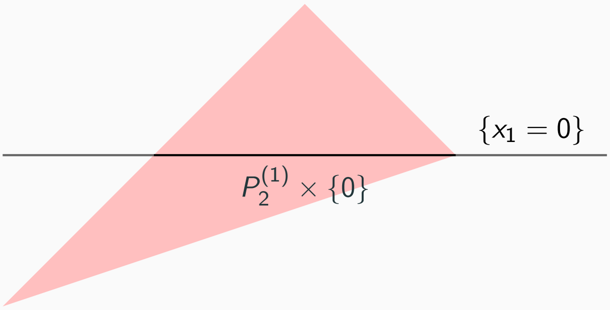
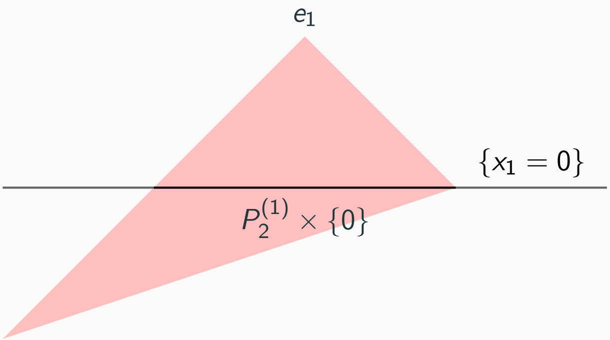


Figure 1: $P_1^{(2)}$ with its cross section $P_2^{(1)} \times \{0\} = P_1^{(2)} \cap \{x_1 = 0\}$.



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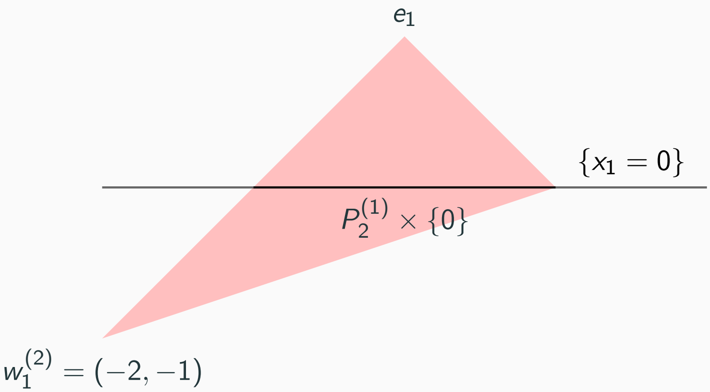

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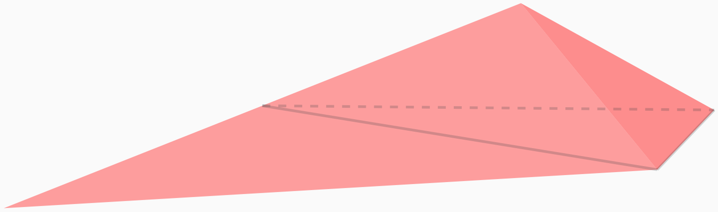
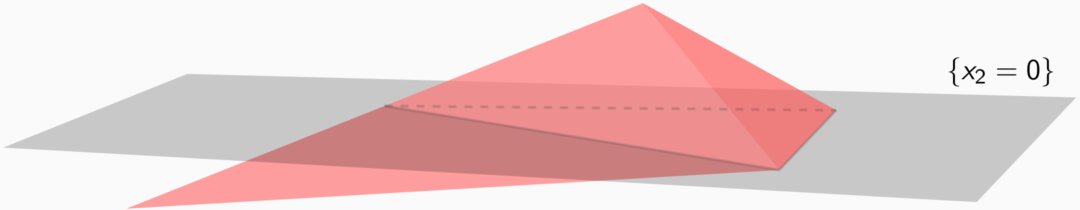


Figure 2: $P_1^{(3)}$



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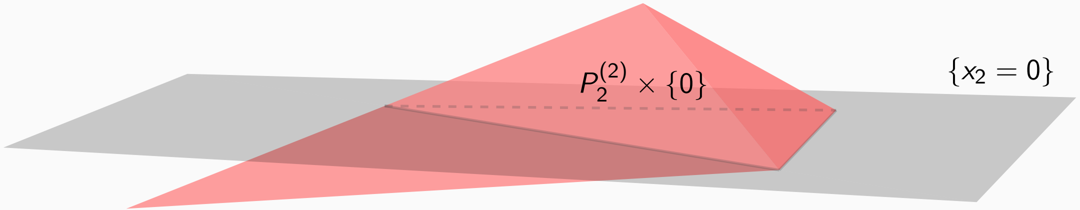

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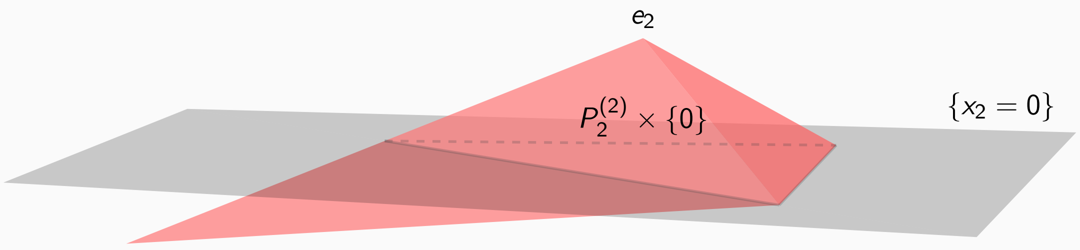

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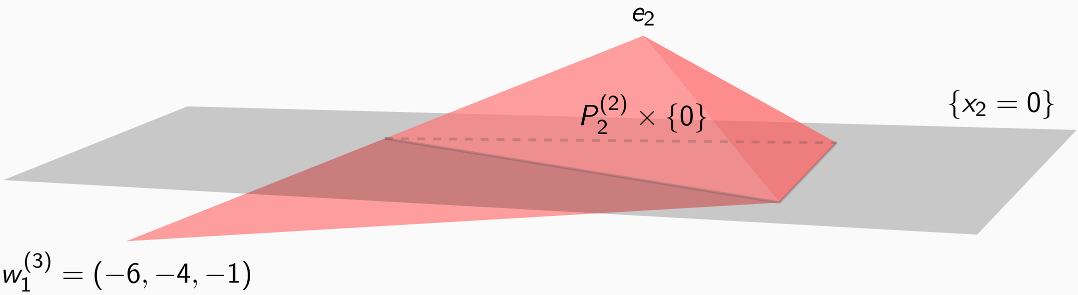

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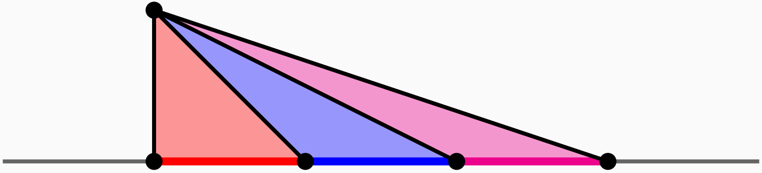

Figure 4: The cone of that triangulation to a new vertex.

Lemma

If $\mathcal{T}$ is unimodular then $\mathcal{T}'$ is unimodular.

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$e_n^{(n+1)}$ and $w_1^{(n+1)}$ are distance 1 from the hyperplane $\{x_n = 0\}$ cutting out $P_2^{(n)}$. $\square$

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If $\mathcal{T}$ is regular then $\mathcal{T}'$ is regular.

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Remark

The inclusion $P_2^{(n-1)} \longrightarrow P_1^{(n)}$ via $x \mapsto (x, 0)$ induces an inclusion $\mathbb{P}_2^{(n-1)} \longrightarrow \mathbb{P}_1^{(n)}$ via $x \mapsto [x : 0]$.

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$P_2^{(n)}$ is isomorphic as a lattice simplex to its polar dual $\check{P}_2^{(n)}$.

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$$\begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ \vdots \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} -1 \\ -1 \\ \vdots \\ s_n - 1 \end{pmatrix}$$

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Furthermore, $\breve{P}_2^{(n-1)} \longrightarrow \breve{P}_2^{(n)}$ via $x \to (x, -1)$ is an isomorphism to the face $\{x_{n-1} = -1\}$.

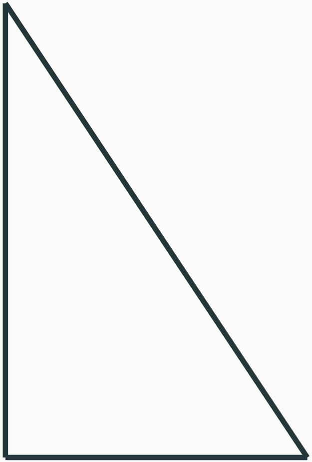
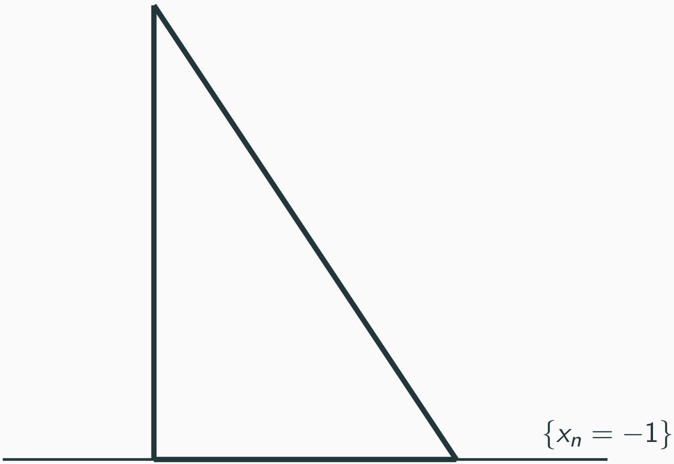


Figure 5: $\check{P}_2^{(n+1)}$



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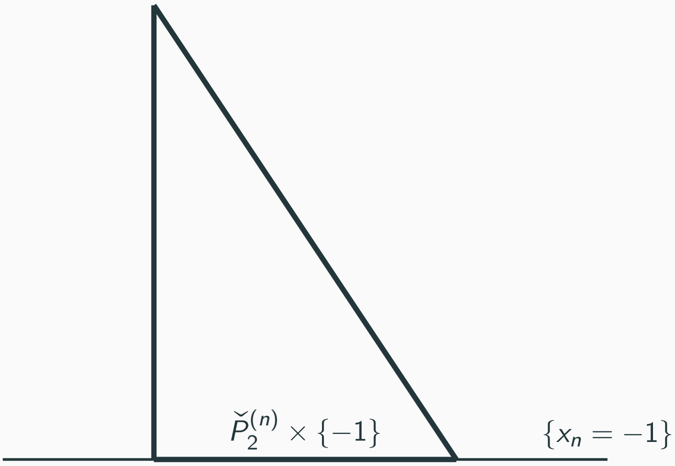

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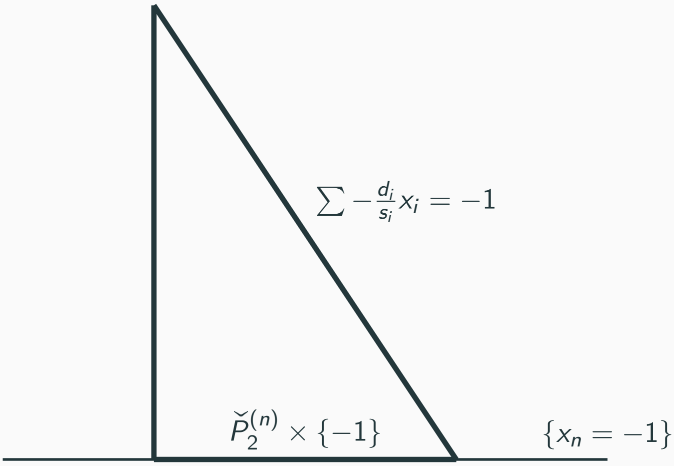

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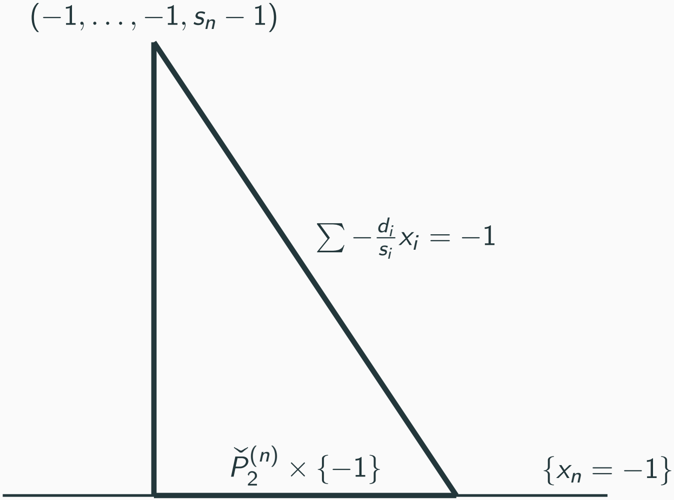

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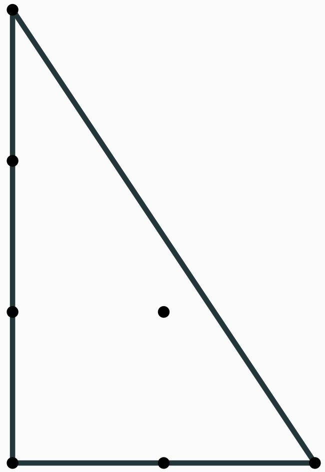
$$\sum_{i=0}^{n-1} \frac{s_n - 1}{s_i} x_i + x_n = 0.$$



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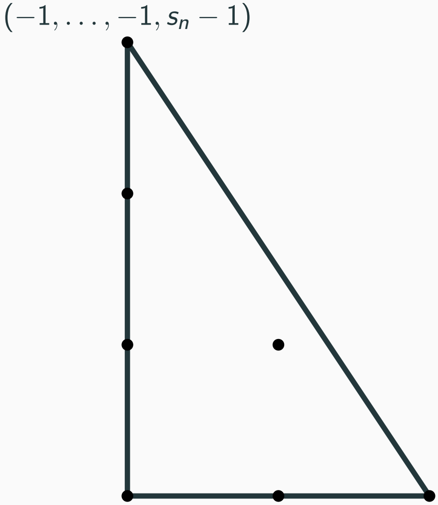

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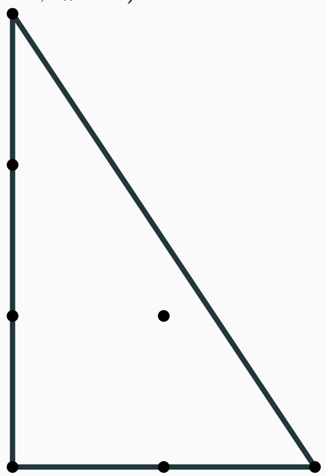

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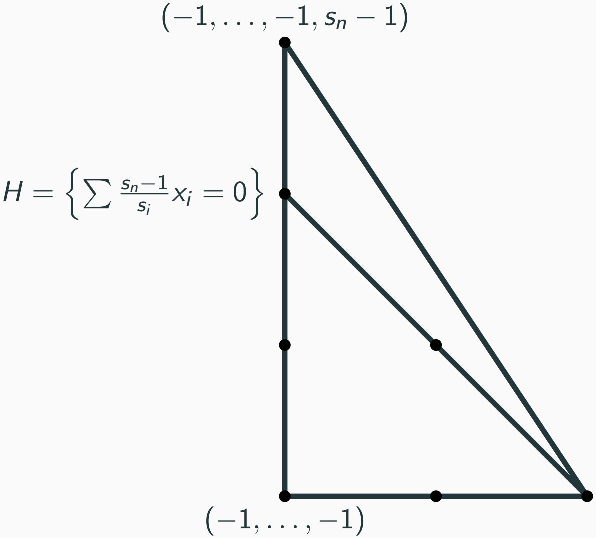

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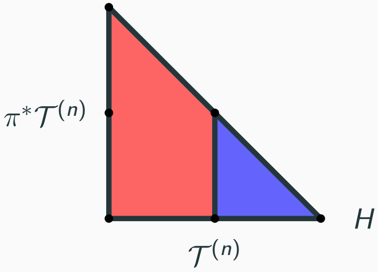

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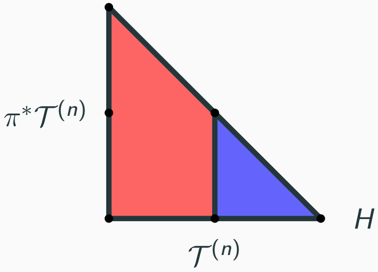

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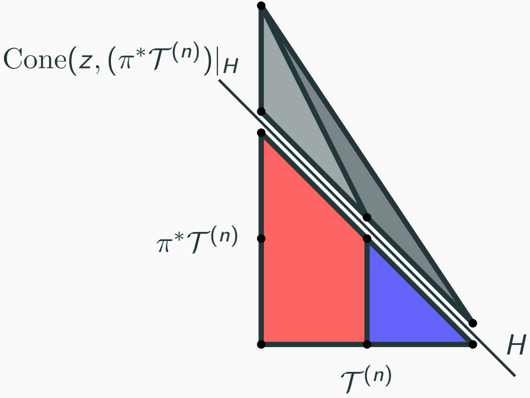

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To triangulate below H, we use the notion of a *pulling refinement*[4].

[4] *Existence of unimodular triangulations - positive results* by Haase, Paffenholz, Piechnik, Santos

Definition (Pulling refinement)

Let $\mathcal{S}$ be a subdivision of a polytope $P \subseteq \mathbb{R}^d$ and $m \in P \cap \mathbb{Z}^d$. The *pulling refinement* $\mathrm{pull}_m(\mathcal{S})$ is defined by replacing every $F \in \mathcal{S}$ containing m by $\mathrm{Conv}(m, F')$ for every face $F' \leq F$ which does *not* contain m.

Lemma

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Lemma

Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ be a projection so that we have $\pi(\mathbb{Z}^{n+1}) = \mathbb{Z}^n$. Let P be a lattice polytope in $\mathbb{R}^{n+1}$ and set $Q = \pi(P)$. Suppose $\mathcal{T}$ is a unimodular triangulation of Q so that $\pi^*\mathcal{T}$ is a lattice subdivision. Let $\mathcal{T}'$ be a refinement of $\pi^*\mathcal{T}$ arising by pulling $\mathcal{S}$ at all lattice points in P in any order. Then $\mathcal{T}'$ is a unimodular triangulation of P.

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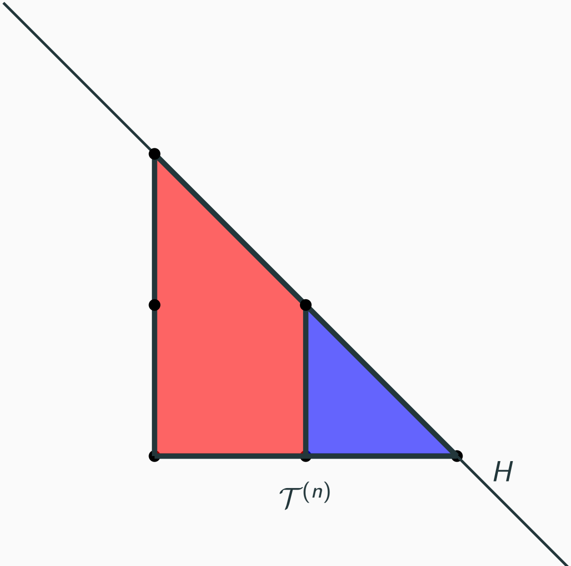

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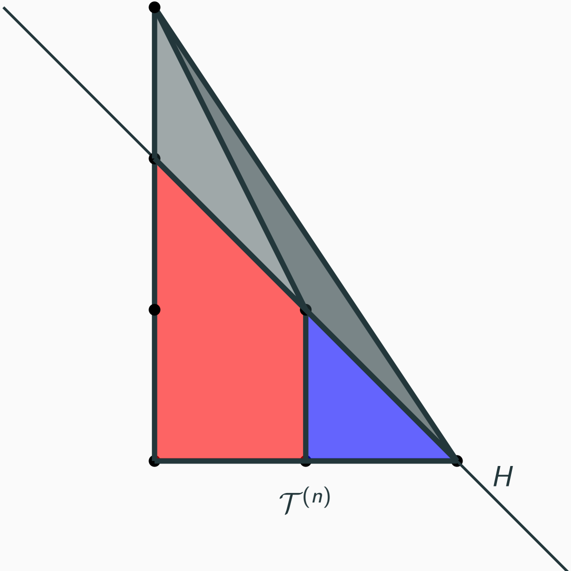

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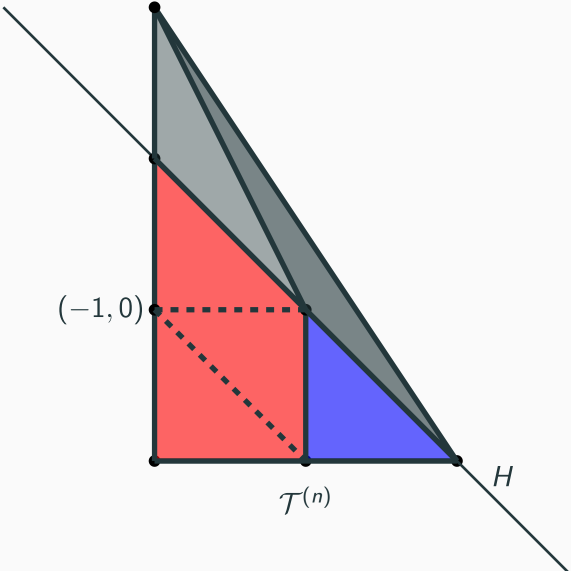

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Resolving the hypersurface

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An equivariant, projective, crepant resolution of X will prove our two main theorems.
We have already constructed a toric, projective, crepant resolution $\pi : \widetilde{\mathbb{P}} \longrightarrow \mathbb{P}$.

Theorem

Let L be the linear system in $\mathbb{P}$ generated by the monomials

$$x_0^{s_0}, \dots, x_{n-1}^{s_{n-1}}, x_n^{d-1} x_{n+1}, x_{n+1}^d.$$

Then for a generic element $X \in L$, $\pi^{-1}X \longrightarrow X$ is a projective, crepant, μ_m-equivariant, resolution of singularities.

Proof idea.

- Compute the base locus of L to be $p = [0 : \cdots : 0 : 1 : 0]$.

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- Show that p is a smooth point of X.

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- Apply Bertini's theorem to show $\pi^{-1}X$ is smooth away from $\widetilde{p} = \pi^{-1}(p)$.
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$\square$

Proof idea.

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- As π is an isomorphism near p, $\widetilde{p}$ is a smooth point of $\pi^{-1}X$.
- Apply a toric automorphism to remove genericity and change X back to $X_1^{(n)}$.

$\square$

Further questions

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$X_3^{(n)}$ is *mirror* to $X_1^{(n)}$ and is the source of Esser, Totaro, and Wang's small volume example, as well as the conjectural largest positive orbifold Euler characteristic